Causal Dynamical Triangulations and Matrix Model
（因果力学的単体分割と行列模型）

Sapporo Winter School   7/1/2009

Titech   Yoshiyuki WATABIKI

MENU

1. DT and CDT [Ambjørn, Loll: hep-th/9805108]
   a. What is DT **(Dynamical Triangulation)** ?
   b. What is CDT **(Causal Dynamical Triangulation)** ?
   c. If we unify the eqs. of DT and CDT, then ・・・
2. 2d CDT and Matrix Model [Ambjørn, Loll, Watabiki, Westra, Zohren: hep-th/0802.0719, 0804.0252, 0810.2408]
   a. A naïve method by matrix model (failed case)
   b. The expression by string field theory
   c. The expression by matrix model
   d. The relation between new matrix model and old one (Wick rotation？)
3. Summary

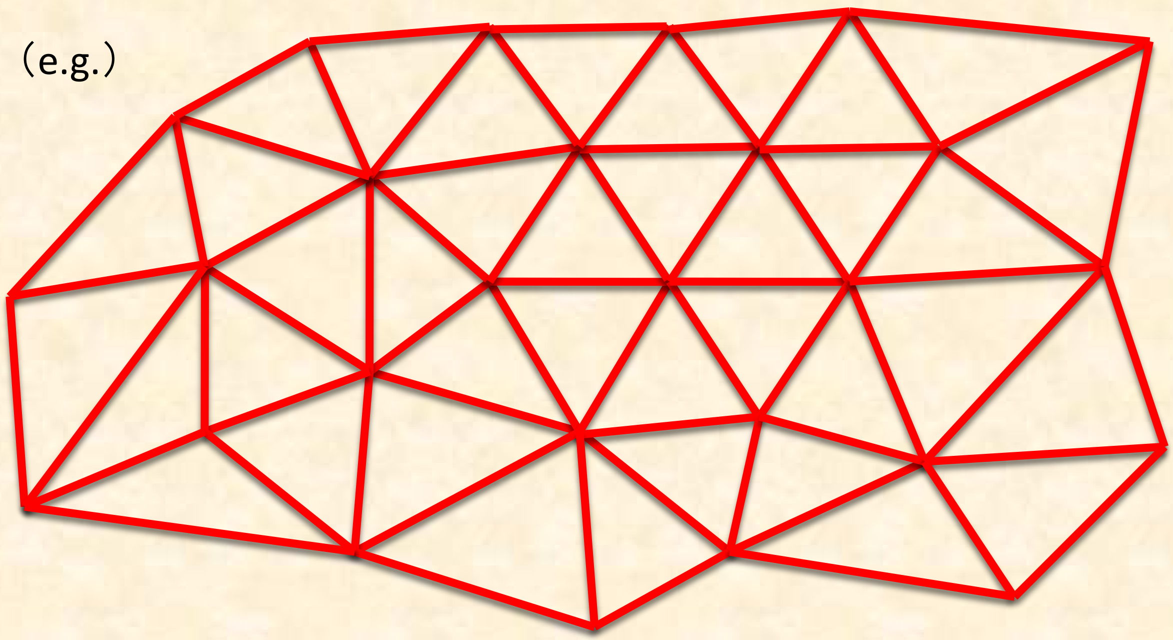
1. DT and CDT

a. What is the DT (Dynamical Triangulation) ?

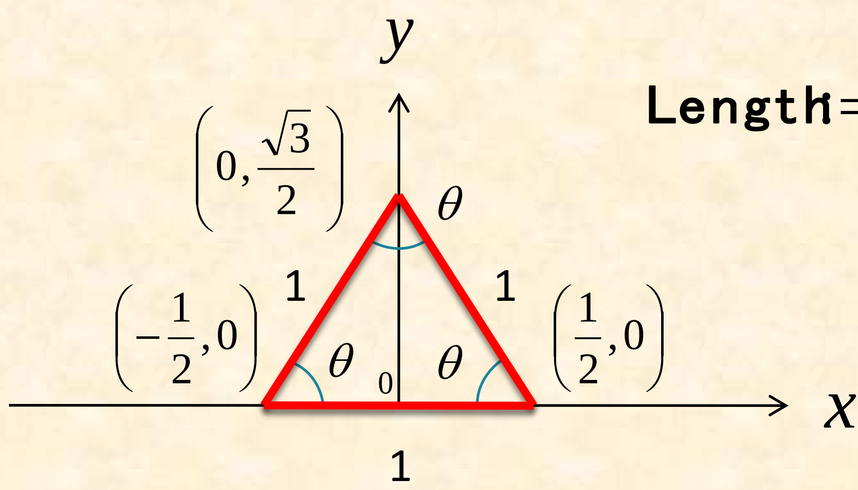
- Definition of 2d DT

Construction of lattice by "equilateral triangles" in 2d Euclidean space

(e.g.)



Each triangle has the same size and equilateral.



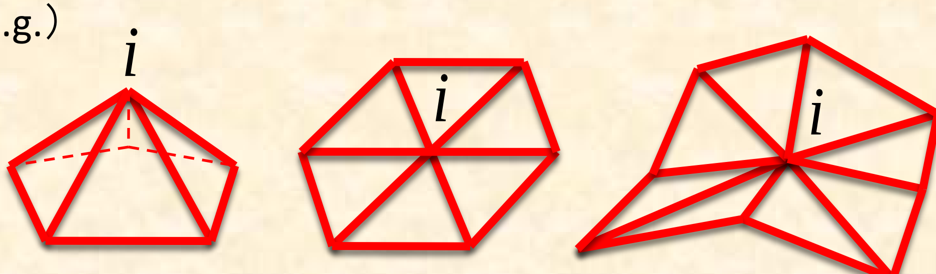
Length $= \sqrt{x^2 + y^2}$

Curvature of site $\boldsymbol{i}$, $\sqrt{g}R_i$ is

(curvature exists only on sites)

$$\sqrt{g}R_i = (6 - q_i)\,\theta$$

(e.g.)



$q_i = 5$	$q_i = 6$	$q_i = 8$
Curv. > 0	Curv.=0	Curv.<0

q_i **is the nr. of triangles together to the site** $\boldsymbol{i}$

- Partition function of 2d DT

Quantum gravity is the path integral of metric $g_{\mu\nu}$
(λ is the cosmological constant)

$$Z = \int Dg_{\mu\nu}\, \mathrm{e}^{\lambda \int \mathrm{d}^2 x \sqrt{g}}$$

$$\begin{cases} \mathrm{e}^{\lambda} \to \kappa \\ \int \mathrm{d}^2 x \sqrt{g} \to N_2 \end{cases}$$

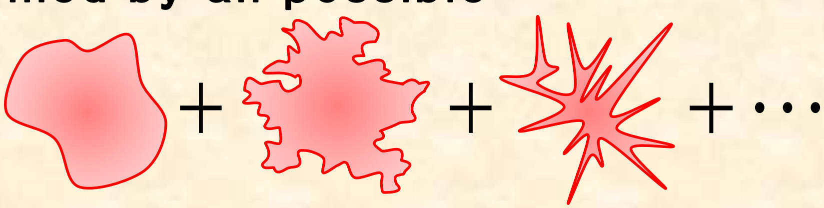
The metric $g_{\mu\nu}$ expresses various curved spaces, so the path integral is the summation of all kinds of triangulated spaces.

(κ is cosmological constant at lattice level, N_2 is nr. of triangles)

One triangle corresponds to one κ

$$Z = \underbrace{\sum \kappa^{N_2}}_{\text{summation of states}}$$

The summation is performed by all possible triangulated lattices by

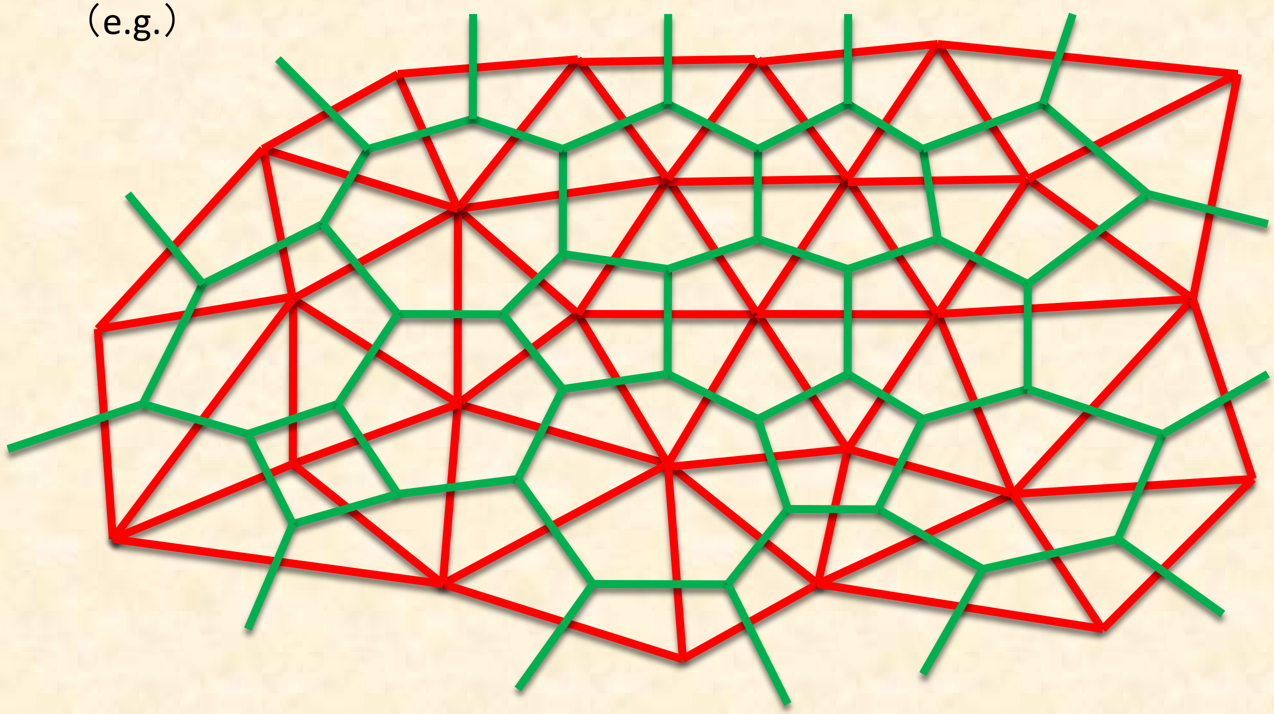


- 2d DT and Matrix model

Definition of dual lattice

connect the c.o.m. of triangles side by side（green lines）

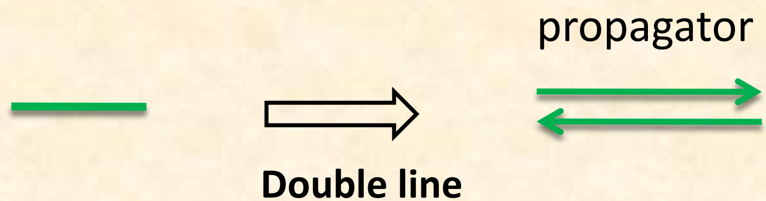
（e.g.）



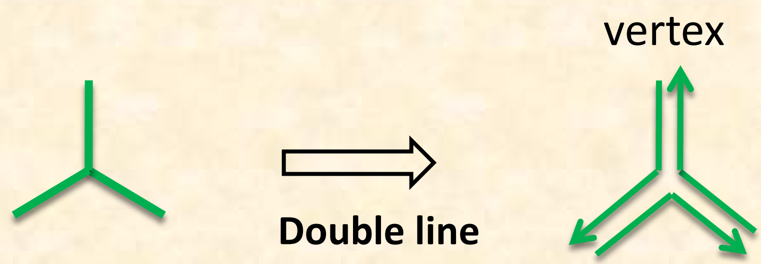
Dual lattice is a Feynman diagram constructed by three point vertices.

Feynman diagrams

Field is expressed by $N \times N$ Hermite matrix φ_i^j



φ^2 $\qquad \dfrac{1}{N}\delta_i^l \delta_j^m$



φ^3 $\qquad \kappa N \delta_i^l \delta_j^m \delta_k^n$

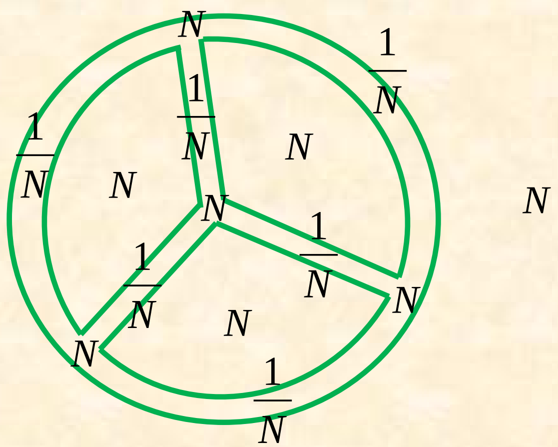
Partition function and Lagrangian

$$Z = \int \mathrm{D}\varphi \, \mathrm{e}^{-NV(\varphi)}$$

$$-NV(\varphi) = \mathrm{tr}\left(-\frac{N}{2}\varphi^2 + \frac{\kappa N}{3}\varphi^3 \right)$$

1- a

Topology and power of N

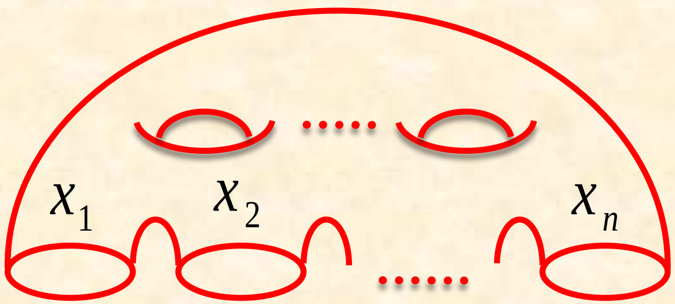


$$N^{N_0}\left(\frac{1}{N}\right)^{N_1} N^{N_2} = N^{N_0-N_1+N_2} = N^{\chi}$$

Amplitude and Laplace transf.

We here introduce the function

$$W(x_1,\ldots,x_n) := N^{n-2}\left\langle \mathrm{tr}\left(\frac{1}{x_1-\varphi}\right)\cdots\mathrm{tr}\left(\frac{1}{x_n-\varphi}\right)\right\rangle_{\text{connected}}$$



W is the partition fun. fixing the topology

(Surface with n holes and several handles)

cf.

$$\frac{1}{x-\varphi} = \frac{1}{x}\frac{1}{1-\varphi/x} = \frac{1}{x}\sum_{\ell=0}^{\infty}\frac{\varphi^{\ell}}{x^{\ell}}$$

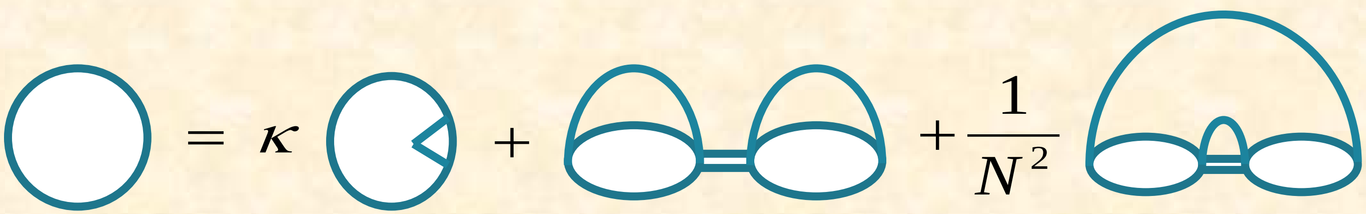
Loop equation

One obtains the loop eq. by $\varphi \rightarrow \varphi + \varepsilon$

$$\int_C \frac{\mathrm{d}z}{2\pi i} \frac{V'(z)}{x-z} W(z) = W^2(x) + \frac{1}{N^2} W(x,x)$$

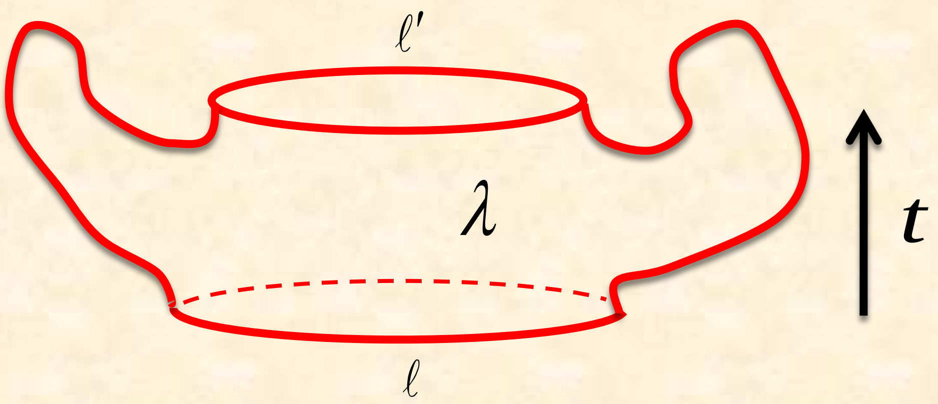
This decides the partition fun of each topology.

Cf: SD eq.



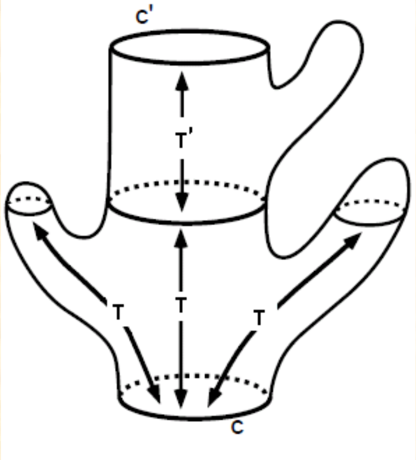
Green関数

Green関数 $G(x, y; t)$ is



and satisfies the decomposition law

$$G(\ell_1, \ell_2; \tau+\tau') = \sum_{\ell} \ell\, G(\ell_1, \ell; \tau)\, G(\ell, \ell_2; \tau')$$



1- a

Laplace transf.

Laplace transf. is defined by

$$G(x,y;t) \;=\; \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty} x^{\ell_1} y^{\ell_2} G(\ell_1,\ell_2;t)$$

and decomposition law becomes

$$G(x,y;\tau+\tau') \;=\; \oint \frac{\mathrm{d}z}{2\pi i z} G(x,\frac{1}{z};\tau)\frac{\mathrm{d}}{\mathrm{d}z} G(z,y;\tau')$$

(**Note**: $\ell \;\leftrightarrow\; -\dfrac{\mathrm{d}}{\mathrm{d}z} \qquad z \;\leftrightarrow\; \dfrac{\mathrm{d}}{\mathrm{d}\ell}$)

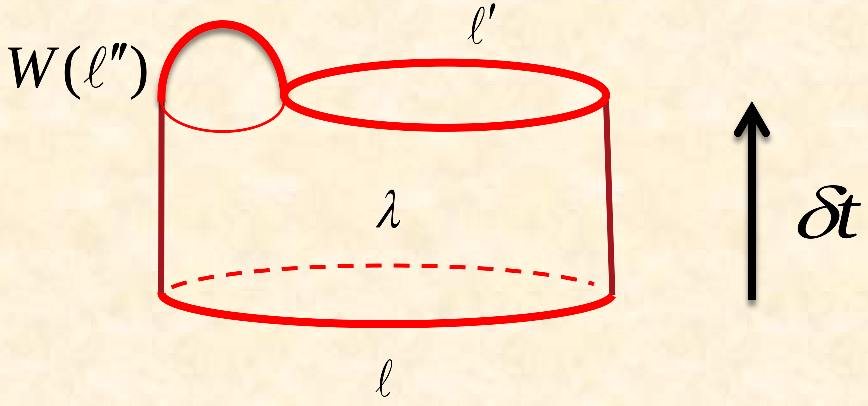
Continuum limit

The continuum limit is obtained by

$$x \;\to\; x_c \mathrm{e}^{-ax} \qquad W \;\to\; (\mathrm{const.}) + a^{\eta} W$$

Then, Green fun of decomposition law becomes

$$\frac{\partial}{\partial t} G(x,y;t) \;=\; -2g \frac{\partial}{\partial x}\Big(W(x) G(x,y;t) \Big)$$

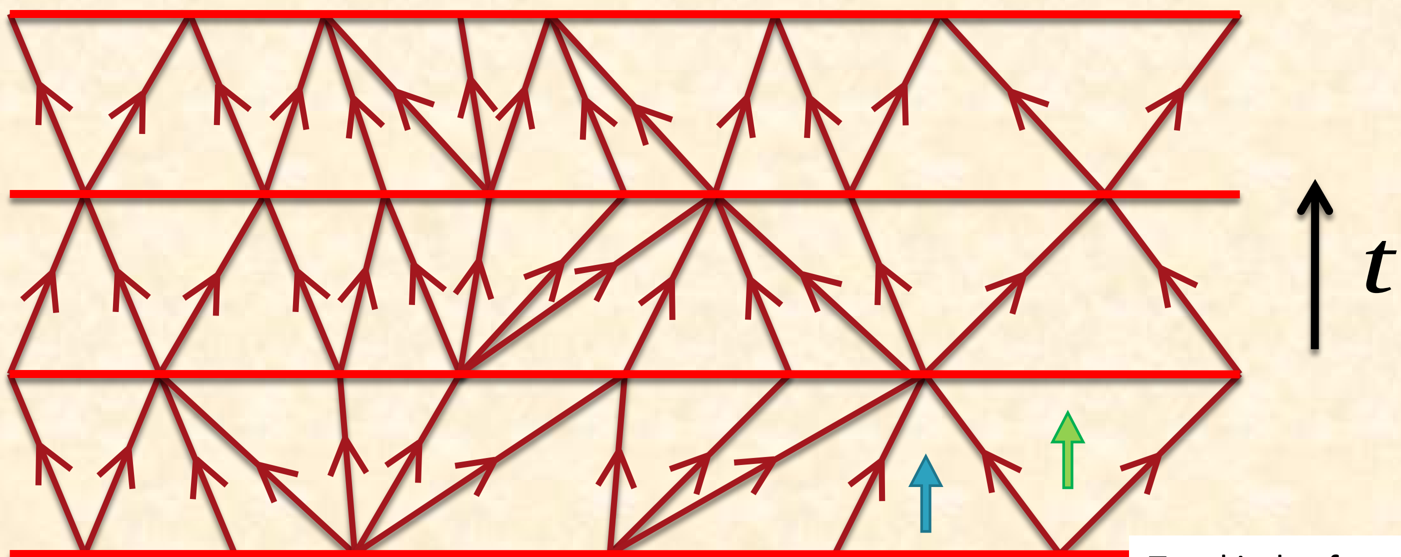


b. What is CDT (Causal Dynamical Triangulation) ?

- Definition of 2d CDT

Construction of lattice by "time (isosceles) triangles" in 2d Minkowski space

(e.g.)

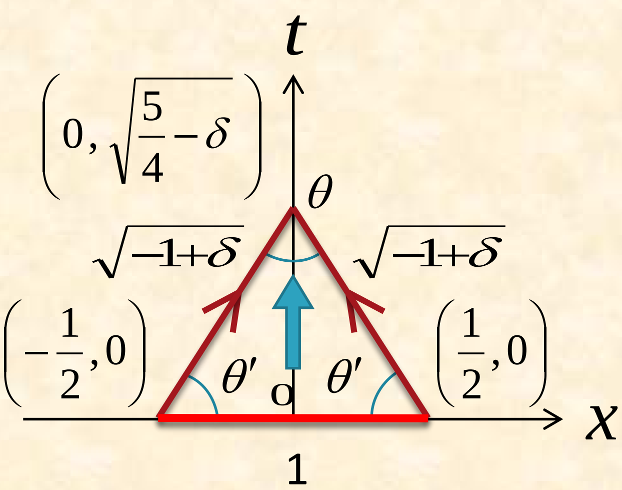


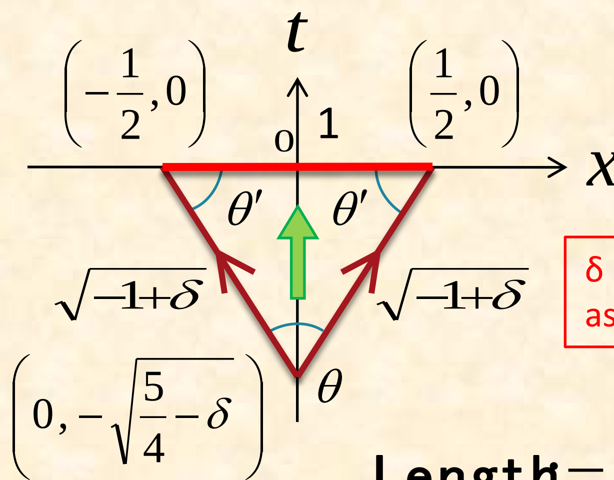
Two kinds of triangle appear.

The direction of time is unique and causal.

(The set of above triangulated lattices is the subset of triangulated lattice by DT.)

Each triangle has the same size and isosceles.



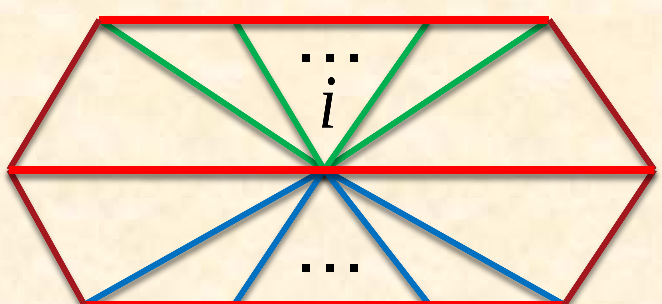


$(\delta < 1)$

δ is related with the asymmetry of *x* and *t*

Length $= \sqrt{x^2 - t^2}$

The curvature of a site i, $\sqrt{g}R_i$ is expressed by



$$\sqrt{g}R_i = (4 - k_i - j_i)\theta$$

k_i Is nr. of green links

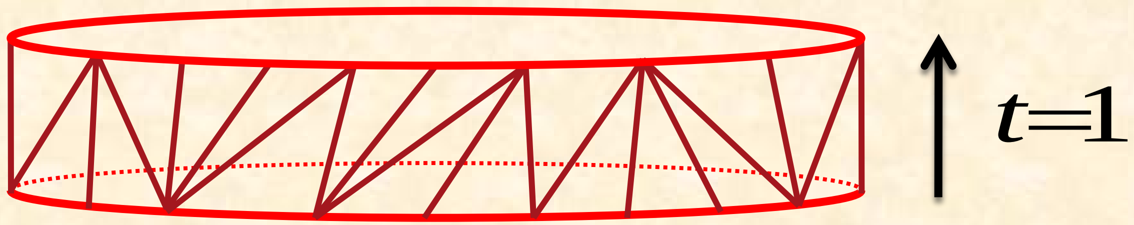
j_i Is nr. of blue links

- 2d CDT and Green fun

Definition of Green fun

The partition fun with cylinder topology is obtained by piling the following lattice

(e.g.)

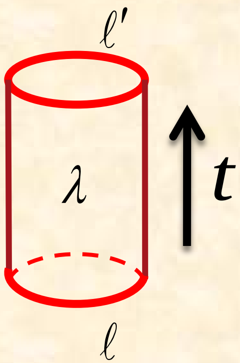


(Green fun)

This partition function is easily obtained because the calculation is a simple summation.

After taking the continuum limit, we obtain the differential eq.

$$\frac{\partial}{\partial t}G(\ell,\ell',t) = \left(-\ell\frac{\partial^2}{\partial\ell^2}+\lambda\ell\right)G(\ell,\ell',t)$$



1- b

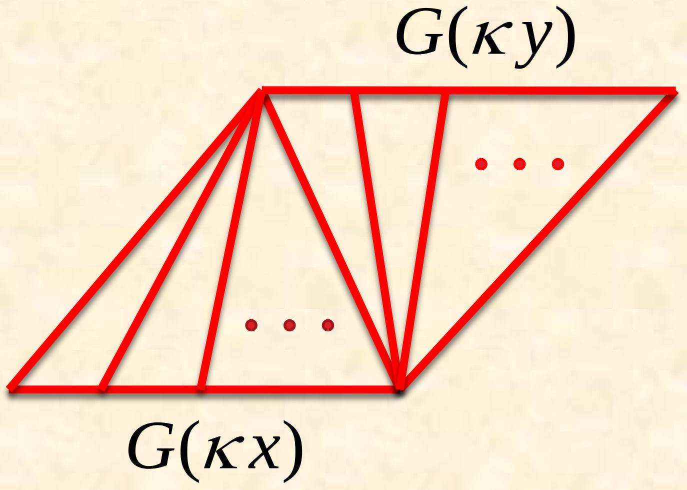
Property of Green fun

The concrete expression of Green **fun is**

$$G(x,y;1) = \sum_{n=1}^{\infty} \frac{1}{n}\left(G(\kappa x)G(\kappa y)\right)^n = -\log\left(1 - G(\kappa x)G(\kappa y)\right)$$

where

$$G(\xi) := \sum_{n=1}^{\infty} \xi^n = \frac{\xi}{1-\xi}$$



Continuum limit of Green fun.

The continuum limit is obtained by

$$t \to \frac{t}{a} \qquad x \to \mathrm{e}^{-ax} \qquad \kappa \to \mathrm{e}^{a^2\lambda}$$

The continuum limit of Green fun

$$G(x,y;t+1) = \oint \frac{\mathrm{d}z}{2\pi i z} G(x,\frac{1}{z};1) \frac{\mathrm{d}}{\mathrm{d}z} G(z,y;t)$$

satisfies the differential eq.

$$\frac{\partial}{\partial t} G(x,y;t) = -\frac{\partial}{\partial x}\Big((x^2-\lambda)G(x,y;t)\Big)$$

The disk amplitude becomes

$$W(x) = \int_0^\infty \mathrm{d}t\, G(x,\ell_2{=}0;t) = \frac{1}{x+\sqrt{\lambda}}$$

C. If we unify the eqs. of DT and CDT, then …

- **Decomposition law of 2d DT and CDT**

If we unify the continuum limit of DT and CDT

$$t \to \frac{t}{a^{\varepsilon}} \qquad x \to x_{\mathrm{c}}\, \mathrm{e}^{-a x} \qquad \kappa \to \kappa_{\mathrm{c}}\, \mathrm{e}^{a^2 \lambda}$$

$$W(x) \to (\mathrm{const.}) + a^{\eta} W(x)$$

Then, the decomposition law becomes

$$a^{\varepsilon} \frac{\partial}{\partial t} G(x, y; t) = -\frac{\partial}{\partial x}\Big(a(x^2 - \lambda) + 2g\, a^{\eta-1} W(x) \Big) G(x, y; t)$$

for DT

$$\varepsilon = \frac{1}{2} \quad \eta = \frac{3}{2}$$

for CDT

$$\varepsilon = 1 \quad \eta = -1$$

Here, only for CDT, we set

$$g \rightarrow a^3 g$$

Then, the decomposition law becomes

$$\frac{\partial}{\partial t} G(x,y;t) = -\frac{\partial}{\partial x}\Big((x^2-\lambda)+2gW(x)\Big)G(x,y;t)$$

At this stage, the unification of both theories is formal. But we consider this unification seriously. In the next section we will construct the theory which leads to the above decomposition law.

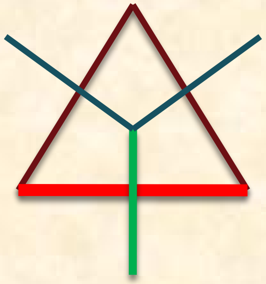
2. 2d CDT and Matrix Model

a. A naïve method by matrix model (failed case)

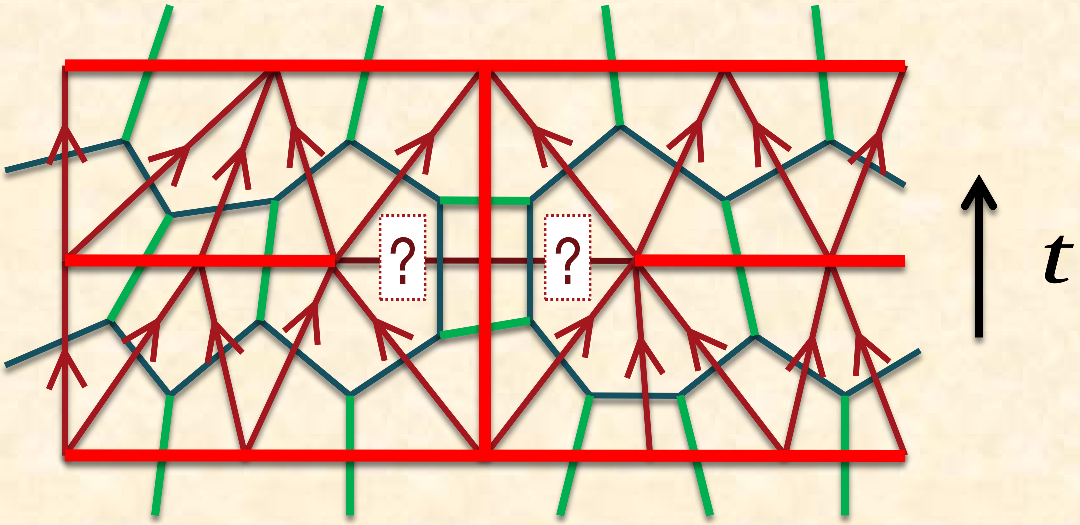
- Use the two kinds of matrix

Let's express the time (isosceles) **triangles**

by $\varphi_{\mathrm{T}}\varphi_{\mathrm{S}}^{2}$



(e.g.)



Time direction of time like links is sometimes not unique!

b. The expression by string field theory

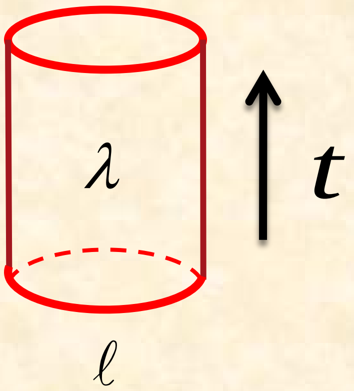
- **Introduction of creation op. and annihilation op.** (ℓ : length)

Creation op. $\Psi^\dagger(\ell)$ **Annihilation op.** $\Psi(\ell)$

$$[\Psi(\ell), \Psi^\dagger(\ell')] = \ell\, \delta_{\ell,\ell'}$$ **(others are zero)**

- **Hamiltonian**

$$H_0 = \int_0^\infty \frac{\mathrm{d}\ell}{\ell} \Psi^\dagger(\ell)\left(-\ell \frac{\partial^2}{\partial \ell^2} + \lambda \ell\right)\Psi(\ell)$$



This differential eq. looks like a diffusion eq.. But the time reversal symmetry is not broken.

Cf.

$$\frac{\partial}{\partial t} G(\ell,\ell',t) = \left(-\ell \frac{\partial^2}{\partial \ell^2} + \lambda \ell\right) G(\ell,\ell',t)$$

$$G(\ell,\ell',t) = \langle 0 | \Psi(\ell') \mathrm{e}^{-tH_0} \Psi^\dagger(\ell) | 0 \rangle$$

Propagation of closed string starting from string with length ℓ

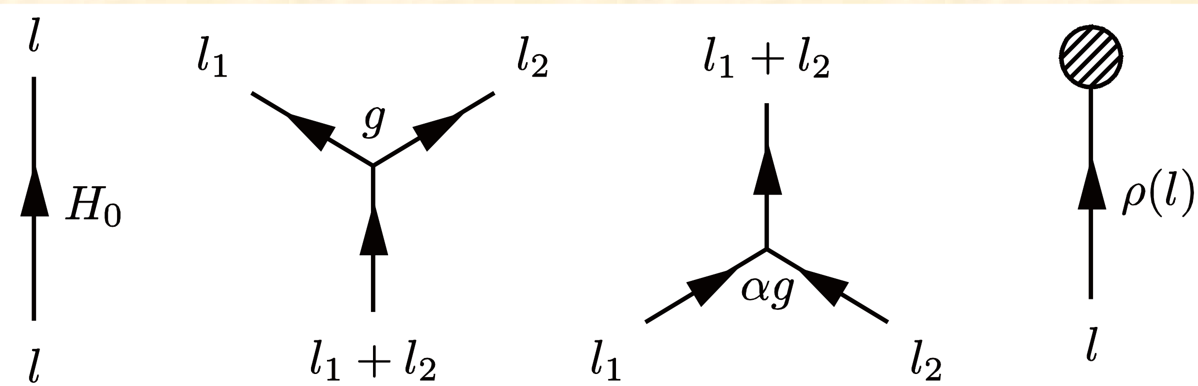
- Hamiltonian which expresses the separation and fusion of strings (universes)

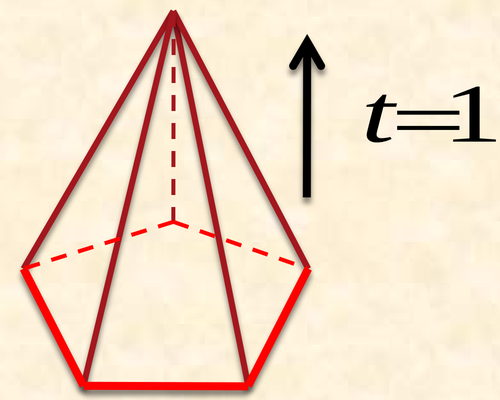
$$H = H_0 - g\int d\ell_1 \int d\ell_2 \Psi^\dagger(\ell_1)\Psi^\dagger(\ell_2)\Psi(\ell_1+\ell_2)$$

$$-\alpha g\int d\ell_1 \int d\ell_2 \Psi^\dagger(\ell_1+\ell_2)\Psi(\ell_1)\Psi(\ell_2)$$

$$-\int \frac{d\ell}{\ell}\rho(\ell)\Psi(\ell)$$

$$\rho(\ell) = \delta(\ell)$$





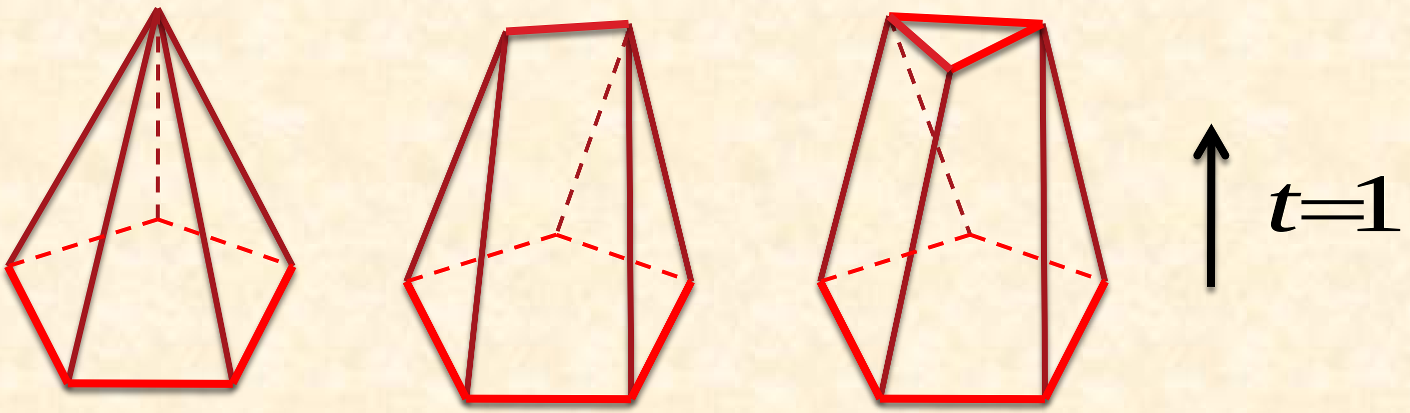
g **is a coupling constant of string theory**

α **is the constant which counts nr. of handles**

ρ(l) of DT is on next page

The case of **DT** :

$$\rho(\ell) = 3\delta''(\ell) - \frac{3\lambda}{4}\delta(\ell)$$



The cosmological constant λ appears.

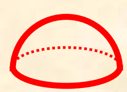
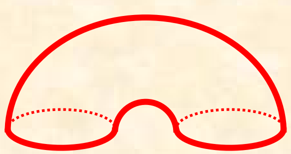
- **Eq. of Schwinger-Dyson**

$$\lim_{t\to\infty}\frac{\partial}{\partial t}\langle 0|\,\mathrm{e}^{-tH}\,\mathrm{e}^{\int \mathrm{d}\ell\, J(\ell)\Psi^{\dagger}(\ell)}|0\rangle \;=\; 0$$

expanding eq. by $J(\ell)$ and perform Laplace transf.

$$W(x) \;=\; \int_0^{\infty}\mathrm{d}\ell\,\frac{1}{x^{\ell+1}}W(\ell) \qquad W(\ell) \;=\; \lim_{t\to\infty}\langle 0|\,\mathrm{e}^{-tH}\,\Psi^{\dagger}(\ell)|0\rangle$$

$$\partial_x\Big(-V'(x)W(x)+W^2(x)+\alpha W(x,x)\Big)-\frac{1}{g} \;=\; 0$$

$W(x)$  $W(x,y)$ 

where

$$V(x) \;=\frac{1}{g}\left(\lambda x-\frac{1}{3}x^3\right) \qquad V'(x) \;=\frac{1}{g}\left(\lambda-x^2\right)$$

for example, for $\alpha = 0$

$$\partial_x\left(-V'(x)W(x)+W^2(x)\right) = \frac{\rho(x)}{g}$$

if one uses

$$W(x)\Big|_{g=0} = \frac{1}{x+\sqrt{\lambda}}$$

then, one obtains

$$\rho(x) = 1 \qquad \textbf{thus,} \qquad \rho(\ell) = \delta(\ell)$$

for higher terms, for example,

$$\partial_x\Big(\{-V'(x)+2W(x)\}W(x,y)+\alpha W(x,x,y)\Big)\ +$$
$$\partial_y\Big(\{-V'(y)+2W(y)\}W(x,y)+\alpha W(x,y,y)\Big)\ +$$
$$2\partial_x\partial_y\frac{W(x)-W(y)}{x-y}=0$$

more higher terms are obtained by the same method

C. Expression by Matrix model

- Matrix model

For the potential $V(\varphi) = \frac{1}{g}\left(\lambda\varphi - \frac{1}{3}\varphi^3\right)$

by the loop eq.

$$\partial_x\left(-V'(x)W(x) + W^2(x) + \frac{1}{N^2}W(x,x)\right) - \frac{1}{g} = 0$$

is obtained.

This eq. coincides with the Schwinger–Dyson eq.

by replacing $\alpha = \frac{1}{N^2}$

Cf.

$$\int_C \frac{dz}{2\pi i}\frac{V'(z)}{x-z}W(z) = W^2(x) + \frac{1}{N^2}W(x,x)$$

- **Loop eq.**

Higher level loop eq. becomes, for example,

$$\partial_x\left(\{-V'(x)+2W(x)\}W(x,y)+\frac{1}{N^2}W(x,x,y)\right) +$$
$$\partial_y\left(\{-V'(y)+2W(y)\}W(x,y)+\frac{1}{N^2}W(x,y,y)\right) +$$
$$2\partial_x\partial_y\frac{W(x)-W(y)}{x-y} = 0$$

more higher level loop eq. are also obtaind by the same way.

These eqs. coincide with the Schwinger-Dyson eqs.

by replacing $\alpha=\dfrac{1}{N^2}$

d. The relation between new matrix model and old one (Wick rotation?)

- Matrix model

Assuming the potential

$$V(\varphi) = \frac{1}{g}\left(-\kappa\varphi + \frac{1}{2}\varphi^2 - \frac{\kappa}{3}\varphi^3\right)$$

then, one obtains the disk amplitude

$$W(x) = \frac{1}{2g}\left(-\kappa + x - \kappa x^2 + \kappa(x-b)\sqrt{(x-c)(x-d)}\right)$$

- Continuum limit

Taking the continuum limit,

disk amplitude at $g=0$

$$W(x) = \frac{1}{x+\sqrt{\lambda}}$$

at $g \neq 0$

$$W(x) = \left(x - \frac{\sqrt{\lambda}}{2}\right)\sqrt{x+\sqrt{\lambda}}$$

where we rescale x **and** λ

SUMMARY

- I introduced CDT **(Causal Dynamical Triangulation) .**
- **I explained the difference between CDT and** DT**.**
- **The potential of 2d** CDT **as a matrix model is**

$$V(\varphi) = \frac{1}{g}\left(\lambda\varphi - \frac{1}{3}\varphi^3\right)$$

- A kind of Wick **rotation is possible**

$$V(\varphi) = \frac{1}{g}\left(-\kappa\varphi + \frac{1}{2}\varphi^2 - \frac{\kappa}{3}\varphi^3\right)$$

g **is a coupling const. of string theory**

κ **is a cosmological constant**